2.3 Relations of Hodge numbers

Let (M, ω) be a compact Kähler manifold. Since $[L, \Delta_{\mathbb{R}}] = [\Lambda, \Delta_{\mathbb{R}}] = 0$, the Lefschetz operator L and its dual Λ induce maps between cohomology groups:

$$L : H^k(M, \mathbb{R}) \to H^{k+2}(M, \mathbb{R}), \quad \Lambda : H^k(M, \mathbb{R}) \to H^{k-2}(M, \mathbb{R}). \tag{2.3.1}$$

Definition 2.3.1. Let (M, ω) be a compact Kähler manifold. Then the primitive cohomology is defined by

$$\begin{aligned} H^k(M, \mathbb{R})_{\text{prim}} &:= \operatorname{Ker}\left(\Lambda : H^k(M, \mathbb{R}) \to H^{k-2}(M, \mathbb{R})\right), \\ H^{p,q}(M)_{\text{prim}} &:= \operatorname{Ker}\left(\Lambda : H^{p,q}(M) \to H^{p-1,q-1}(M)\right). \end{aligned} \tag{2.3.2}$$

Note that the primitive cohomology does not depend on the chosen Kähler structure and only on the cohomology class of the Kähler form $[\omega] \in H^{1,1}(M)$.

Theorem 2.3.2 (Hard Lefschetz Theorem)**.** *Let (M, ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} M = n$. Then for $k \le n$,*

$$L^{n-k} : H^k(M, \mathbb{R}) \simeq H^{2n-k}(M, \mathbb{R}) \tag{2.3.3}$$

and for any k,

$$H^k(M, \mathbb{R}) = \bigoplus_{i \ge (k-n)_+} L^i H^{k-2i}(X, \mathbb{R})_{\text{prim}}, \tag{2.3.4}$$

where $a_+ = \max\{a, 0\}$. Moreover, these two isomorphisms are compatible with the bidegree decomposition. It means that for $k \le n$,

$$L^{n-k} : H^{p,k-p}(M) \simeq H^{n+p-k,n-p}(M), \tag{2.3.5}$$

and for any k,

$$H^{p,q}(M) = \bigoplus_{i \ge (p+q-n)_+} L^i H^{p-i,q-i}(X)_{\text{prim}}. \tag{2.3.6}$$

In particular,

$$H^k(M, \mathbb{R})_{\text{prim}} = \bigoplus_{p+q=k} H^{p,q}(M)_{\text{prim}}. \tag{2.3.7}$$

In order to prove the Hard Lefschetz theorem, we need the following lemma.

Lemma 2.3.3. *For $\alpha \in \Lambda^k T^*M$, we have*

$$[L,\Lambda]\alpha = (k-n)\alpha \tag{2.3.8}$$

and

$$[L^i,\Lambda]\alpha = i(k-n+i-1)L^{i-1}\alpha. \tag{2.3.9}$$

Proof. We prove (2.3.8) by induction. If $\dim_{\mathbb{C}} M = 1$, for $\alpha \in \Lambda^0 T^*M$, $[L,\Lambda]\alpha = -\Lambda L\alpha = -\alpha$; for $\alpha \in \Lambda^1 T^*M$, $[L,\Lambda]\alpha = 0$; for $\alpha \in \Lambda^2 T^*M$, $[L,\Lambda]\alpha = L\Lambda\alpha = \alpha$. (2.3.8) holds.

Assume that (2.3.8) holds for $\dim_{\mathbb{C}} M = m$. If $\dim_{\mathbb{C}} M = m+1$, for $x \in M$, we split T_xM by $T_xM = U \oplus V$ such that $\dim_{\mathbb{R}} V = 2$. Then $\Lambda^k T^*M = \oplus_{i=0}^2 \Lambda^{k-i}U \otimes \Lambda^i V$. For $\alpha \in \Lambda^k T^*M$, $\alpha = \beta_0 \otimes \beta_0' + \beta_1 \otimes \beta_1' + \beta_2 \otimes \beta_2'$. Thus, for $j = 0,1,2$,

$$\begin{aligned}[L,\Lambda]\beta_j \otimes \beta_j' &= L\left(\Lambda(\beta_j)\otimes\beta_j' + \beta_j \otimes \Lambda(\beta_j')\right) - \Lambda\left(L(\beta_j)\otimes\beta_j' + \beta_j\otimes L(\beta_j')\right)\\ &= [L,\Lambda](\beta_j)\otimes\beta_j' + \beta_j\otimes[L,\Lambda](\beta_j') = (k-j-m)\beta_j\otimes\beta_j' + (j-1)\beta_j\otimes\beta_j'\\ &= (k-m-1)\beta_j\otimes\beta_j'. \end{aligned} \tag{2.3.10}$$

Therefore, we get (2.3.8).

We also prove (2.3.9) by induction. By (2.3.8), (2.3.9) holds for $i = 1$. Assume that (2.3.9) holds for $i = m$. For $i = m+1$,

$$\begin{aligned}[L^{m+1},\Lambda]\alpha &= L^{m+1}\Lambda\alpha - \Lambda L^{m+1}\alpha = L[L^m,\Lambda]\alpha + [L,\Lambda]L^m\alpha\\ &= m(k-n+m-1)L^m\alpha + (2m+k-n)L^m\alpha\\ &= (m+1)(k-n+m)L^m\alpha. \end{aligned} \tag{2.3.11}$$

Therefore, we get (2.3.9).

The proof of our lemma is completed. □

The Hard Lefschetz theorem Theorem 2.3.2 follows directly from $[L,\Delta_{\mathbb{R}}] = 0$ and the following proposition.

Proposition 2.3.4. *Let $P^k = \{\alpha \in \Lambda^k T^*M : \Lambda\alpha = 0\}$.*

(i) If $u \in P^k$, then $L^s u = 0$ for $s \geq (n-k+1)_+$.

(ii) If $k > n$, then $P^k = 0$.

(iii) The map $L^{n-k} : \Lambda^k T^*M \to \Lambda^{2n-k}T^*M$ is bijective.

(iv) If $k \leq n$, then $P^k = \{\alpha \in \Lambda^k T^*M : L^{n-k+1}\alpha = 0\}$.

(v) There exists orthogonal decomposition $\Lambda^k T^*M = \oplus_{i \geq (k-n)_+} L^i(P^{k-2i})$.

Proof. For (i), by (2.3.9), for $u \in P^k$,

$$\Lambda^s L^r u = \Lambda^{s-1}(\Lambda L^r - L^r \Lambda)u = r(n-k-r+1)\Lambda^{s-1}L^{r-1}u. \tag{2.3.12}$$

By induction, for $r \geq s$, we have

$$\Lambda^s L^r u = r(r-1)\cdots(r-s+1)\cdot(n-k-r+1)\cdots(n-k-r+s)L^{r-s}u. \tag{2.3.13}$$

Take $r = n+1$. Then $L^r u = 0$. Thus

$$(n+1)\cdots(n-s+2)\cdot(-k)\cdots(-k+s-1)L^{r-s}u = 0. \tag{2.3.14}$$

So if $s \leq k$, we have $L^{n+1-s}u = 0$, which is equivalent to (i).

Take $s = 0$ in (i). We get (ii).

(iii) Since $\mathrm{rk}(\Lambda^k T^*M) = \mathrm{rk}(\Lambda^{2n-k}T^*M)$, we only need to prove the injectivity. We prove it by induction on k. For $k = 0$, L^n is injective. We assume that the injectivity holds for $k \leq m-1$. For $k = m$, $r \leq n-k$, we can assume that L^{r-1} is injective on $\Lambda^m T^*M$. For $\alpha \in \Lambda^m T^*M$, if $L^r\alpha = 0$, then by Lemma 2.3.3,

$$\begin{aligned} &L^{r-1}(L\Lambda - r(m-n+r-1)\,\mathrm{Id})\alpha \\ &\qquad = [L^r, \Lambda]\alpha - r(m-n+r-1)L^{r-1}\alpha = 0. \end{aligned} \tag{2.3.15}$$

Thus $(L\Lambda - r(m-n+r-1)\,\mathrm{Id})\alpha = 0$. Since $r \leq n-m$, $r(m-n+r-1) \neq 0$. Thus there exists $\beta \in \Lambda^{m-2}T^*M$, such that $\alpha = L\beta$ and $L^{r+1}\beta = 0$. Since L^{r+1} is injective, $\beta = 0$. So $\alpha = 0$ and L^r is injective. By induction, we get L^m is injective. So (iii) holds for any $k \leq n$.

(iv) If $\alpha \in P^k$, from (ii), we have $L^{n-k+1}\alpha = 0$. If $L^{n-k+1}\alpha = 0$, we have $L^{n-k+1}\Lambda\alpha = 0$. Since L^{n-k+2} is bijective, we have $\Lambda\alpha = 0$.

(v) is equivalent to the statement that for any $\alpha \in \Lambda T^*M$, there exists unique decomposition

$$\alpha = \sum_{r \geq (k-n)_+} L^r u^r, \quad u_r \in P^{k-2r}. \tag{2.3.16}$$

We first study the uniqueness. Assume that $\alpha = 0$ and there exists r such that $u_r \neq 0$. Let s be the largest integer such that $u_s \neq 0$. Then

$$\Lambda^s\alpha = 0 = \sum_{(k-n)_+ \leq r \leq s} \Lambda^s L^r u_r = \sum_{(k-n)_+ \leq r \leq s} \Lambda^{s-r}\Lambda^r L^r u_r = \sum_{(k-n)_+ \leq r \leq s} c_{k,r}\Lambda^{s-r}u_r, \tag{2.3.17}$$

where $c_{k,r} = r!(n-k+r+1)\cdots(n-k+2r)$ by (2.3.13). Since $\Lambda u_r = 0$ for any r, we have $u_s = 0$, a contradiction.

For the existence, by (iii), we can assume that $k \le n$ and prove it by induction on k. It is obvious for $k = 1$. We assume that (2.3.16) exists for any $k < m$. By (iii), for $\alpha \in \Lambda^m T^*M$ there exists $\beta \in \Lambda^{m-2}T^*M$, such that $L^{n-m+2}\beta = L^{n-m+1}\alpha$. Let $\alpha_0 = \alpha - L\beta$. Then $L^{n-m+1}\alpha_0 = 0$. From (iv), $\alpha_0 \in P^m$.

The proof of our proposition is completed. □

Let $b^k(M) := \dim_{\mathbb{C}} H^k(M,\mathbb{C})$ be the usual **Betti number** of M and let $h^{p,q}(M) := \dim_{\mathbb{C}} H^{p,q}(M)$ be the so-called **Hodge number** of M when M is a complex manifold. Remark that b^k is a topological invariant but $h^{p,q}$ might be not, which depends on the complex structure.

In this section, we assume that (M,ω) is a compact Kähler manifold.

By Theorem 2.2.25 and (2.2.100), we have

$$\begin{aligned}
&b^k = \sum_{p+q=k} h^{p,q}; && \Leftarrow H^k(M,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(M);\\
&h^{p,q} = h^{q,p}; && \Leftarrow H^{p,q}(M) \simeq \overline{H^{q,p}(M)};\\
&h^{p,q} = h^{n-p,n-q}; && \Leftarrow \text{Serre duality:} H^{p,q}(M) \simeq H^{n-p,n-q}(M)^*;\\
&h^{p,q} = h^{n-q,n-p}; && \Leftarrow \text{Hodge}* : H^{p,q}(M) \simeq H^{n-q,n-p}(M);\\
&h^{p,k-p} = h^{n+p-k,n-p}, \forall p \le k \le n, && \Leftarrow \text{Hard Lefschetz} : H^{p,k-p}(M) \simeq H^{n+p-k,n-p}(M).
\end{aligned} \tag{2.3.18}$$

A popular picture to describe the relations is the **Hodge diamond**:

$$\begin{array}{lccccccccl}
b^0 & & & & & h^{0,0} & & & & \\
b^1 & & & & h^{1,0} & & h^{0,1} & & & \\
b^2 & & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & & & & & \cdots & & & & \\
b^n & h^{n,0} & & & & \textbf{Serre} \searrow & & & h^{0,n} & \updownarrow \textbf{Hodge} \\
 & & & & & \cdots & & & & \\
b^{2n-2} & & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
b^{2n-1} & & & & h^{n,n-1} & & h^{n-1,n} & & & \\
b^{2n} & & & & & h^{n,n} & & & & \\
 & & & & & \overset{\longleftrightarrow}{\textbf{Conj}}. & & & &
\end{array}$$

Theorem 2.3.5. *On compact Kähler manifolds, we have*

(1) the odd Betti numbers b^{2k+1} are even;

(2) $h^{1,0} = \frac{1}{2}b^1$ is a topological invariant;

(3) the even Betti numbers b^{2k} are positive;

(4) $h^{p,p}$ are positive.

(5) if $k = p + q \le n$, then $h^{p,q} \ge h^{p-1,q-1}$, $b_k \ge b_{k-2}$; If $k = p + q \ge n$, then $h^{p,q} \ge h^{p+1,q+1}$, $b_k \ge b_{k+2}$.

Proof. The first statement follows from

$$b^{2k+1} = \sum_{p=0}^{2k+1} h^{p,2k+1-p} = 2\sum_{p=0}^{k} h^{p,2k+1-p} \tag{2.3.19}$$

(2) is obvious.

For (3), if $\omega^k = d\alpha$, by Stokes' theorem, $\int_M \omega^n = \int_M d(\alpha \wedge \omega^{n-k}) = 0$. It will not happen since by (2.1.58), ω^n is a volume form. So ω^k is d-closed and not d-exact.

For (4), we observe that $\omega^p \in \Omega^{p,p}(M)$ and is $\bar{\partial}$-exact. If $\omega^p = \bar{\partial}\beta$, then $\omega^n = \bar{\partial}(\beta \wedge \omega^{n-p})$ is $\bar{\partial}$-exact. But $[\omega]^n \in H^{2n}(M, \mathbb{C}) \simeq H^{n,n}(M)$ is not equal to 0 since it is a volume form. So ω^p is not $\bar{\partial}$-exact.

For (5), let $h^{p,q}_{\text{prim}} = \dim H^{p,q}(M)_{\text{prim}}$. Then Theorem 2.3.2 says that if $p + q \le n$,

$$h^{p,q} = h^{p,q}_{\text{prim}} + h^{p-1,q-1}_{\text{prim}} + \cdots \tag{2.3.20}$$

and if $p + q \ge n$,

$$h^{p,q} = h^{n-q,n-p}_{\text{prim}} + h^{n-q-1,n-p-1}_{\text{prim}} + \cdots . \tag{2.3.21}$$

So we get (5).

The proof of our theorem is completed. □

Corollary 2.3.6. *The only sphere that admits a Kähler structure is S^2.*

Let $P^{p,q}_{\mathbb{C}} = \{\alpha \in \Lambda^p T^{(1,0)*}M \otimes \Lambda^q T^{(0,1)*}M : \Lambda\alpha = 0\}$.

Lemma 2.3.7. *For $\alpha \in P^{p,q}_{\mathbb{C}}$, $p + q = k$, we have*

$$*\alpha = (-1)^{\frac{k(k+1)}{2}} \sqrt{-1}^{p-q} \frac{L^{n-k}\alpha}{(n-k)!}. \tag{2.3.22}$$

Proof. We only need to prove it at one point of M. In this proof we regard $T^{(1,0)*}M$ as $\mathbb{C}^n$. Let $dz_1, \cdots dz_n$ be a basis. For $S = \{i_1, \cdots, i_s\}$, we denote by $\omega_S = \left(\frac{\sqrt{-1}}{2}\right)^s dz_{i_1} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge dz_{i_s} \wedge d\bar{z}_{i_s}$. We can write

$$\alpha = \sum_{A,B,S} \gamma_{A,B,S} dz_A \wedge dz_M \wedge \omega_S, \tag{2.3.23}$$

where A, B, S are disjoint subsets of $\{1, \cdots, n\}$. Let $\alpha_{A,B} = \sum_S \gamma_{A,B,S} dz_A \wedge d\bar{z}_B \wedge \omega_S$. Thus $\Lambda\alpha = 0$ implies that $\Lambda\alpha_{A,B} = 0$. So we only need to prove the lemma for

$$\alpha = dz_A \wedge d\bar{z}_B \wedge \sum_S \gamma_S \omega_S. \tag{2.3.24}$$

In this sum, we only need to consider the subsets $S \subset K := \{1, \cdots, n\} - (A \cup B)$ and the cardinal $m = |S| = (k - |A| - |B|)/2$. Since $\Lambda\alpha = 0$, for any $N \subset K$ with $|N| = m - 1$, we have

$$\sum_{i \in K - N} \gamma_{N \cup \{i\}} = 0. \tag{2.3.25}$$

Let ${}^cS$ be the complement of S in K. Then by (2.2.30), we have

$$\begin{aligned}(d\bar{z}_A \wedge dz_B \wedge \omega_S) \wedge *(dz_A \wedge d\bar{z}_B \wedge \omega_S) &= \text{vol} \\ &= \left(\frac{\sqrt{-1}}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.\end{aligned} \tag{2.3.26}$$

After a careful calculation, we have

$$*(dz_A \wedge d\bar{z}_B \wedge \omega_S) = (-1)^{m + \frac{k(k+1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^{n-k} \sqrt{-1}^{p-q} dz_A \wedge d\bar{z}_B \wedge \omega_{{}^cS} \tag{2.3.27}$$

So

$$*\alpha = \sum_{S \subset K} (-1)^{m + \frac{k(k+1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^{n-k} \sqrt{-1}^{p-q} \gamma_S dz_A \wedge d\bar{z}_B \wedge \omega_{{}^cS}. \tag{2.3.28}$$

On the other hand,

$$\begin{aligned}&(-1)^{\frac{k(k+1)}{2}} \sqrt{-1}^{p-q} \frac{L^{n-k}\alpha}{(n-k)!} \\ &\quad = (-1)^{\frac{k(k+1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^{n-k} \sqrt{-1}^{p-q} \sum_{S,N} \gamma_S dz_A \wedge d\bar{z}_B \wedge \omega_{S \cup N} \\ &= \sum_{J \subset K} (-1)^{\frac{k(k+1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^{n-k} \sqrt{-1}^{p-q} \left(\sum_{S \subset J} \gamma_S\right) dz_A \wedge d\bar{z}_B \wedge \omega_J,\end{aligned} \tag{2.3.29}$$

where N runs through the subsets of cardinal $n - k$ contained in K and disjoint from S.

For every $r \le m$, let $S_r = \sum_{|N\cap J|=r} \gamma_N$. Then $S_0 = \gamma_{^cJ}$ and $S_m = \sum_{N\subset J}\gamma_N$. Then (2.3.25) implies that $(r+1)S_{r+1} = -(m-r)S_r$. So $S_m = (-1)^m S_0$. It means that

$$\sum_{S\subset J} \gamma_S = (-1)^m \gamma_{^cJ}. \tag{2.3.30}$$

From (2.3.28), (2.3.29) and (2.3.30), we get (2.3.22).

The proof of our lemma is completed. □

Let (M,ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} M = n$. The Poincaré duality implies a non-degenerate pairing

$$\langle\ ,\ \rangle : H^k(M,\mathbb{R}) \times H^{n-k}(M,\mathbb{R}) \to \mathbb{R}. \tag{2.3.31}$$

We define the intersection form Q on $H^k(M,\mathbb{R})$, $k \le n$ by

$$Q(\alpha,\beta) = \langle L^{n-k}\alpha, \beta\rangle = \int_M \omega^{n-k}\wedge\alpha\wedge\beta. \tag{2.3.32}$$

Clearly, it is symmetric for k even and antisymmetric for k odd. Thus on $H^k(M,\mathbb{C})$, the sesquilinear form

$$H_k(\alpha,\beta) = (\sqrt{-1})^k Q(\alpha,\bar{\beta}) \tag{2.3.33}$$

is a Hermitian form.

Lemma 2.3.8. *For $k \le n$, the Lefschetz decomposition*

$$H^k(M,\mathbb{C}) = \bigoplus_{i\ge 0} L^i H^{k-2i}(X,\mathbb{C})_{\mathrm{prim}}. \tag{2.3.34}$$

is orthogonal for H^k. Moreover, on each primitive component $L^iH^{k-2i}(X,\mathbb{C})_{\mathrm{prim}}$, H_k induces the form $(-1)^i H_{k-2i}$.

Proof. For $\alpha = L^r\alpha'$, $\beta = L^s\beta'$, with α', β' primitive and $r < s$, we have $L^{n-k}\alpha\wedge\beta = L^{n-k+r+s}\alpha'\wedge\beta'$. By Proposition 2.3.4 (iv), we have $L^{n-k+r+s}\alpha' = 0$. Thus $H_k(\alpha,\beta) = 0$. The second statement is obvious.

The proof of our lemma is completed. □

The curve case of the following theorem is due to Riemann.

Theorem 2.3.9 (Hodge-Riemann bilinear relation)**.** *Let (M,ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} M = n$. The decomposition $H^k(M,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(M)$ is orthogonal for H_k. Moreover, the form $(\sqrt{-1})^{p-q-k}(-1)^{\frac{k(k+1)}{2}} H_k$ is positive definite on $H^{p,q}(M)_{\mathrm{prim}}$.*

Proof. The first statement follows directly by counting the degrees.

For the second statement, for $\alpha \in H^{p,q}(M)_{\mathrm{prim}}$ the harmonic form, by Lemma 2.3.7,

$$\begin{aligned} H_k(\alpha,\alpha) &= (\sqrt{-1})^k \int_M \alpha \wedge L^{n-k}\bar{\alpha} \\ &= (\sqrt{-1})^k (n-k)!(-1)^{\frac{k(k-1)}{2}} \sqrt{-1}^{q-p} \int_M \alpha \wedge *\bar{\alpha} \\ &= (n-k)!(-1)^{\frac{k(k-1)}{2}} \sqrt{-1}^{q-p+k} \|\alpha\|_{L_2}^2. \end{aligned} \tag{2.3.35}$$

The proof of our theorem is completed. □

Theorem 2.3.10 (Hodge index theorem)**.** *Let (M,ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} M = n$ even. Let $\mathrm{sign}(Q)$ be the signature of the intersection form $Q(\alpha,\beta) = \int_M \alpha \wedge \beta$ on $H^n(M,\mathbb{R})$. Then*

$$\mathrm{sign}(Q) = \sum_{p,q} (-1)^p h^{p,q}. \tag{2.3.36}$$

In particular, the number $\sum_{p,q}(-1)^p h^{p,q}$ is a topological invariant.

Proof. For $n = 2k$, $\alpha \in H^{p,q}(M)_{\mathrm{prim}}$, $p+q = n-2r$, we have

$$\mathrm{sign}(Q) = (-1)^k \mathrm{sign}(H). \tag{2.3.37}$$

By (2.3.35),

$$\begin{aligned} H(L^r\alpha) = (-1)^r H_{n-2r}(\alpha) &= (-1)^r(-1)^{k-r-p}(2r)!\|\alpha\|_{L^2}^2 \\ &= (2r)!(-1)^{k+p}\|\alpha\|_{L^2}^2. \end{aligned} \tag{2.3.38}$$

So

$$\mathrm{sign}(Q) = \sum_{r\geq 0, p+q=n-2r} (-1)^p h^{p,q}_{\mathrm{prim}} = \sum_{p+q=n} (-1)^p \sum_{j\geq 0} (-1)^j h^{p-j,q-j}_{\mathrm{prim}}. \tag{2.3.39}$$

By (2.3.20), we have $h^{p,q}_{\mathrm{prim}} = h^{p,q} - h^{p-1,q-1}$. So

$$\begin{aligned} \mathrm{sign}(Q) &= \sum_{p+q=n} (-1)^p \left(h^{p,q} + 2\sum_{j>0} (-1)^j h^{p-j,q-j} \right) \\ &\overset{(1)}{=} \sum_{p+q=n} (-1)^p \left(h^{p,q} + \sum_{j\neq 0} (-1)^j h^{p-j,q-j} \right) \\ &= \sum_{p+q \text{ even}} (-1)^p h^{p,q} \overset{(2)}{=} \sum_{p,q} (-1)^p h^{p,q}. \end{aligned} \tag{2.3.40}$$

Here (1) uses the Serre duality and (2) follows from $(-1)^p h^{p,q} + (-1)^q h^{q,p} = 0$ if $p + q$ is odd.

The proof of our theorem is completed. □

Definition 2.3.11. Let M be a compact complex manifold of dimension n. The Hirzebruch χ_y-genus is the polynomial

$$\chi_y := \sum_{p,q=0}^{n} (-1)^q h^{p,q} y^p. \tag{2.3.41}$$

It is a special case of the elliptic genus, a mathematical analogue of the partition function in physics. The following theorem is the corollary of the Hirzebruch-Riemann-Roch Theorem 2.1.29.

Theorem 2.3.12. *In local terms,*

$$\chi_y = \int_M \mathrm{Td}(T^{1,0}M) \left(\sum_{p=0}^{n} y^p \,\mathrm{ch}(T^{(p,0)*}M) \right). \tag{2.3.42}$$

If $y = 0$, $\chi_0 = \sum_{q=0}^{n} (-1)^q h^{0,q}$ and $\mathrm{Td}(M) := \int_M \mathrm{Td}(T^{1,0}M)$ are two definitions of the arithmetic genus in the history.

If $y = 1$, and if M is Kähler with even complex dimension, then $\chi_1 = \mathrm{sign}(Q)$ in Theorem 2.3.10. In this case, (2.3.42) reads

$$\mathrm{sign}(Q) = \int_M L(M), \tag{2.3.43}$$

where L is defined in (2.1.87). This is the Hirzebruch signature theorem, which also holds for compact $4k$-dimensional manifolds.

If $y = -1$, and if M is Kähler,

$$\chi_{-1} = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q} = \sum_{k=0}^{n} (-1)^k b_k = e(M), \tag{2.3.44}$$

the Euler number. In this case, Theorem 2.3.12 means that

$$e(M) = \int_M c_n(M) = c_n(M). \tag{2.3.45}$$

This is the Gauss-Bonnet-Chern Theorem for complex manifolds. Note that (2.3.45) also holds on compact complex manifolds.

We finish this chapter by the famous Hodge conjecture.

Definition 2.3.13. The fundamental class $[Z] \in H^{p,p}(M)$ of a complex submanifold $Z \subset M$ of codimension p in M is defined by

$$\int_M \alpha \wedge [Z] = \int_Z \alpha|_Z \tag{2.3.46}$$

for any $\alpha \in H^{2n-2p}(M)$.

Definition 2.3.14. If M is a complex submanifold of a complex projective space, then M is called a projective manifold.

Now we could state a version of the Hodge conjecture.

Conjecture 2.3.15 (Hodge conjecture)**.** Let M be a projective manifold. For any $\alpha \in H^{p,p}(M) \cap H^{2p}(M, \mathbb{Q})$, it could be generated linearly by the fundamental classes with coefficients in $\mathbb{Q}$.

Remark that the Hodge conjecture is false for Kähler manifolds. And there exists $\alpha \in H^{p,p}(M) \cap H^{2p}(M, \mathbb{Z})$ such that it could not be generated linearly by the fundamental classes with coefficients in $\mathbb{Z}$.

Here we summarize the supercommutative relations of $\partial, \bar{\partial}, \partial^*, \bar{\partial}^*, L$ and Λ for compact Kähler manifold, which contains the Kähler identity.

Let $[A, B] = AB - (-1)^{|A||B|}BA$.

B \ A	∂	$\bar{\partial}$	∂^*	$\bar{\partial}^*$	L	Λ
∂	∂^2	0	Δ	0	0	$\sqrt{-1}\bar{\partial}^*$
$\bar{\partial}$	0	$\bar{\partial}^2$	0	Δ	0	$-\sqrt{-1}\partial^*$
∂^*	Δ	0	$\partial^{*,2}$	0	$\sqrt{-1}\bar{\partial}$	0
$\bar{\partial}^*$	0	Δ	0	$\bar{\partial}^{*,2}$	$-\sqrt{-1}\partial$	0
L	0	0	$-\sqrt{-1}\bar{\partial}$	$\sqrt{-1}\partial$	0	$n-k$
Λ	$-\sqrt{-1}\bar{\partial}^*$	$\sqrt{-1}\partial^*$	0	0	$k-n$	0